Circuits, Signals, and Systems

William McC. Siebert

The MIT Press
Cambridge, Massachusetts London, England

McGraw-Hill Book Company
New York St. Louis San Francisco Montreal Toronto

This book is one of a series of texts written by faculty of the Electrical Engineering and Computer Science Department at the Massachusetts Institute of Technology. It was edited and produced by The MIT Press under a joint production-distribution arrangement with the McGraw-Hill Book Company.

Ordering Information:

North America
Text orders should be addressed to the McGraw-Hill Book Company.
All other orders should be addressed to The MIT Press.

Outside North America
All orders should be addressed to The MIT Press or its local distributor.

This book was set under the direction of the author using the TEX typesetting system and was printed and bound by Halliday Lithograph in the United States of America.

Library of Congress Cataloging in Publication Data

Siebert, William McC.
Circuits, signals, and systems.
(The MIT electrical engineering and computer science series)
Includes index.
1. Electric circuits. 2. Discrete-time systems.
3. Linear time invariant systems. I. Title. III. Series.
TK454.S57 1985 621.319'2 85-4302
ISBN 0-262-19229-2 (MIT Press)
ISBN 0-07-057290-9 (McGraw-Hill)

16

DURATION-BANDWIDTH RELATIONSHIPS AND THE UNCERTAINTY PRINCIPLE

16.0 Introduction

The step response of a typical lowpass filter will have nonzero *rise time* and nonzero *delay*, as illustrated in Figure 16.0–1. Since the step response is the integral of the impulse response, it should be clear that any measure of step response rise time is also a measure of impulse response *duration*, and vice versa. The principal purpose of this chapter is to show that the impulse response duration (or step response rise time) is inversely related to the frequency response bandwidth. For any particular shape of impulse response, it is obvious from dimensional considerations or the scale-change property of Table XIII.2 that such an inverse relationship should exist. The remarkable thing is that—independent of shape—there is a lower bound on the duration-bandwidth product; this is the formal content of the *Uncertainty Principle*, which will be our main analytical result.

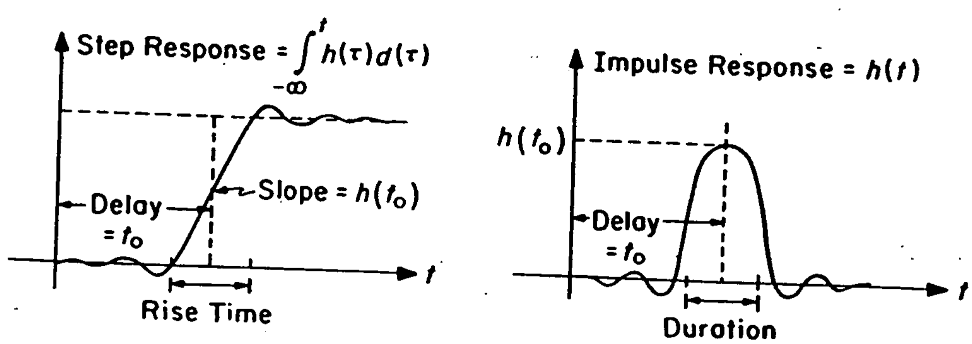

Figure 16.0–1. Delay, rise time, and duration.

16.1 Definitions of Delay, Rise Time, Duration, and Bandwidth

The quantities of interest to us can be formally defined in an unlimited number of ways. No one way, or set of ways, is best for all purposes. For example, a direct, easily interpreted measure of rise time is the time it takes the step response to go from 10% to 90% of its final value. But such a measure is tedious to compute

and clumsy to manipulate. On the other hand, a very simple and convenient definition of rise time would be the ratio of the final value of the step response to the slope of the step response at some appropriate point along the rise (such as the 50% point). If we call the time at this point t_0, this definition of rise time can be expressed in a simple formula,

$$\text{Rise time} = \frac{\int_{-\infty}^{\infty} h(t)\,dt}{h(t_0)} \tag{16.1–1}$$

where $h(t)$ as usual represents the impulse response and we have exploited the fact that the step response is the integral of the impulse response. We can also understand this formula as defining the "width" of $h(t)$ as the ratio of its "area" to its "height." But (16.1–1) can give unreasonable values for the rise time if $h(t)$ is not a single, narrow, largely positive pulse-like waveform such as that shown in Figure 16.0–1. (See, for example, Problem 16.1.)

If we are to relate the rise time to the bandwidth, we must also come up with a way of specifying the effective bandwidth of an arbitrary lowpass frequency response $H(f)$. Except for the fact that $H(f)$ is in general complex, this problem is mathematically identical with specifying the duration of $h(t)$. A potential ambiguity is introduced by the conjugate symmetry of $H(f)$ (assuming that $h(t)$ is real). Thus, when we specify the bandwidth of $H(f)$, do we wish to include all frequencies, $-\infty < f < \infty$ (the bandwidth of the *double-sided* spectrum) or simply the positive frequencies, $f > 0$ (the bandwidth of the *single-sided* spectrum)? For example, should we specify the bandwidth of the ideal lowpass filter

$$H(f) = \begin{cases} 1, & |f| < W \\ 0, & \text{elsewhere} \end{cases}$$

as W or $2W$? Since little more than a factor of 2 is involved, this would seem to be largely a matter of style and convenience. Unfortunately, there is no uniform approach to this matter in the literature; considerable care (and sometimes a Ouija board) may be necessary to discover each author's intent. We shall not be dogmatic in this book either, but shall endeavor to make our intentions clear whenever ambiguity might arise.

Examples of various ways to measure durations and bandwidths—some fairly generally applicable and others restricted to the particular example in which they are applied—are shown in Figure 16.1–1. Note that in each case the product of duration and bandwidth is on the order of unity (within a factor of 2 or so). This is the observation we wish to generalize and make more precise.

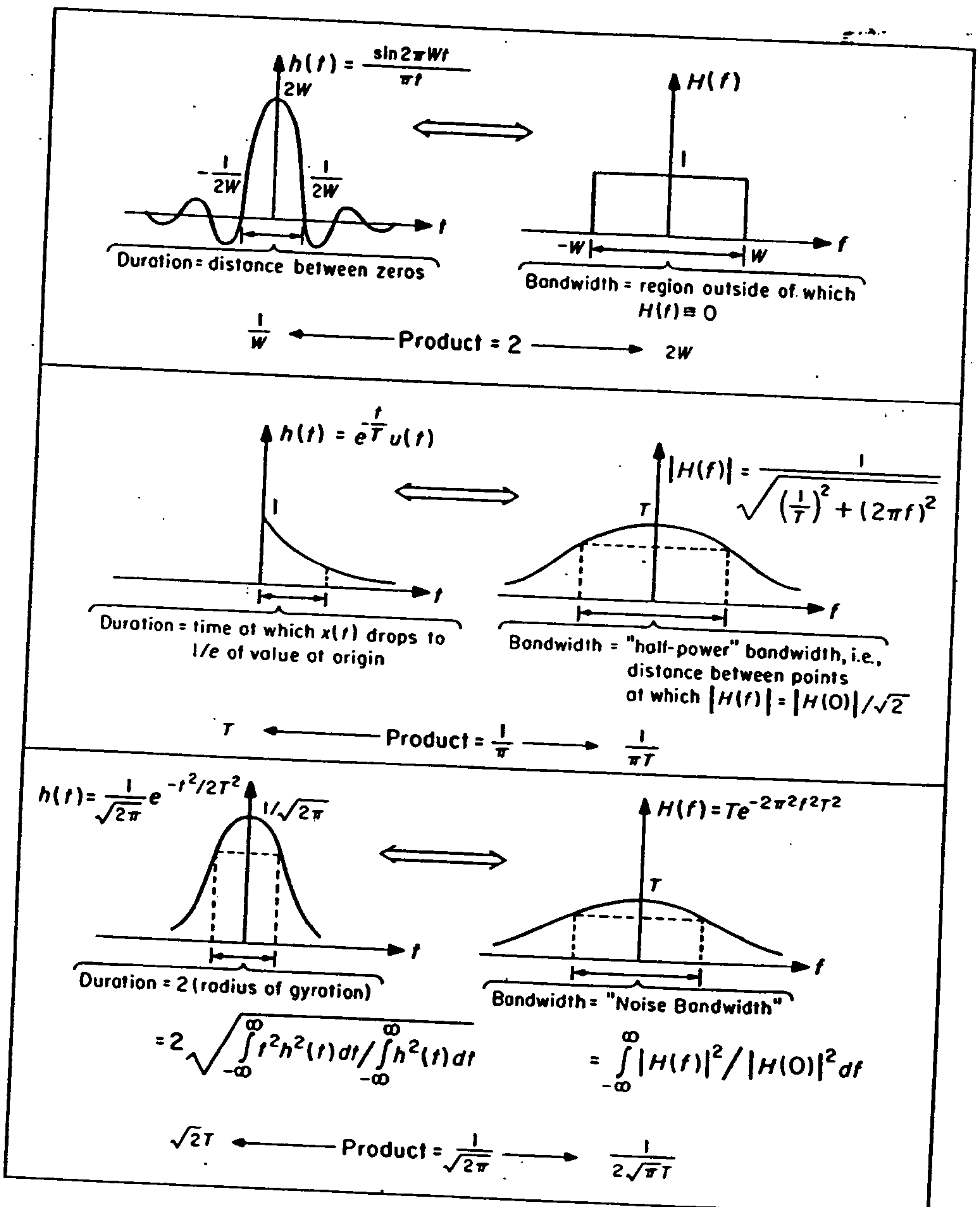

Figure 16.1–1. Examples of various measures of duration and bandwidth.

Probably the most analytically useful definitions of the delay, duration, and bandwidth are given by various moments of $h(t)$ and $H(f)$, or even better of $h^2(t)$ and $|H(f)|^2$. The definitions and some important properties of these moments are given in the following examples.*

Example 16.1-1

Consider an amplifier made up of cascaded stages. The stages need not be identical. Let $h_i(t)$ be the impulse response of the i^{th} stage and assume that $h_i(t) \geq 0$, so that the step response of each stage (or indeed of any number of cascaded stages) is monotonic. Assuming that both integrals exist, define

$$T_i = \frac{\int_{-\infty}^{\infty} t h_i(t)\,dt}{\int_{-\infty}^{\infty} h_i(t)\,dt}. \tag{16.1-2}$$

T_i, the (normalized) *first moment* of $h_i(t)$, can be interpreted as the center of gravity of a mass distributed along the t-axis with density $h_i(t)$. Thus we may consider T_i as measuring the delay in the impulse or step response of the i^{th} stage.

It is then easy to show that the delay resulting from a cascade of n stages is the sum of the delays of each stage; that is, if

$$h(t) = h_1(t) * h_2(t) * \cdots * h_n(t) \tag{16.1-3}$$

then

$$T = T_1 + T_2 + \cdots + T_n\,. \tag{16.1-4}$$

In particular, if the stages are identical, the delay is directly proportional to the number of stages. We shall prove this for a cascade of two stages; the general result then follows by induction. Combining several formulas from Table XIII.2, we deduce that

$$\int_{-\infty}^{\infty} t h(t)\,dt = \frac{-1}{j2\pi}\frac{dH(f)}{df}\bigg|_{f=0}. \tag{16.1-5}$$

If

$$h(t) = h_1(t) * h_2(t) \tag{16.1-6}$$

then

$$H(f) = H_1(f)H_2(f) \tag{16.1-7}$$

and

$$\int_{-\infty}^{\infty} t h(t)\,dt = \frac{-1}{j2\pi}\left[H_2(0)\frac{dH_1(f)}{df}\bigg|_{f=0} + H_1(0)\frac{dH_2(f)}{df}\bigg|_{f=0}\right]. \tag{16.1-8}$$

Moreover,

$$\int_{-\infty}^{\infty} h(t)\,dt = H(0) = H_1(0)H_2(0). \tag{16.1-9}$$

*It is perhaps worth pointing out that, although we shall state all the results of this chapter in terms of system impulse responses and their transforms, $h(t)$ and $H(f)$, the results obviously apply broadly to any time function and its transform.

Finally,

$$T = \frac{\int_{-\infty}^{\infty} th(t)\,dt}{\int_{-\infty}^{\infty} h(t)\,dt} = -\frac{1}{j2\pi}\left[\frac{dH_1(f)/df|_{f=0}}{H_1(0)} + \frac{dH_2(f)/df|_{f=0}}{H_2(0)}\right]$$
$$= T_1 + T_2 \tag{16.1–10}$$

as was to be shown.

Similarly, assuming that the integrals exist, we may define

$$(\Delta T_i)^2 = 4\left[\frac{\int_{-\infty}^{\infty} t^2 h_i(t)\,dt}{\int_{-\infty}^{\infty} h_i(t)\,dt} - T_i^2\right]$$
$$= 4\frac{\int_{-\infty}^{\infty}(t-T_i)^2 h_i(t)\,dt}{\int_{-\infty}^{\infty} h_i(t)\,dt}. \tag{16.1–11}$$

$(\Delta T_i/2)^2$ is just the (normalized) moment of inertia about the center of gravity of a mass distribution $h_i(t)$.* That is, ΔT_i is twice the radius of gyration of the mass distribution, and thus a measure of the duration of $h_i(t)$ or of the rise time of the step response of the i^{th} stage. Again we can prove (see Problem 16.3) that for a cascade of n stages,

$$(\Delta T)^2 = (\Delta T_1)^2 + (\Delta T_2)^2 + \cdots + (\Delta T_n)^2. \tag{16.1–12}$$

Thus, in particular, for identical stages the rise time is proportional to the *square root* of the number of stages.

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If $h_i(t)$ is not positive, the duration measure of (16.1–11) becomes somewhat dubious. A better measure is obtained by first squaring $h(t)$:

$$(\Delta T)^2 = 4\left[\frac{\int_{-\infty}^{\infty} t^2 h^2(t)\,dt}{\int_{-\infty}^{\infty} h^2(t)\,dt} - \left(\frac{\int_{-\infty}^{\infty} th^2(t)\,dt}{\int_{-\infty}^{\infty} h^2(t)\,dt}\right)^2\right]. \tag{16.1–13}$$

Indeed, in many ways ΔT of (16.1–13) is the most analytically satisfactory simple general measure of duration; for virtually any $h(t)$ for which the integrals exist, (16.1–13) will give a not unreasonable estimate of the duration. Equivalently,

$$(\Delta W)^2 = 4\frac{\int_{-\infty}^{\infty} f^2|H(f)|^2\,df}{\int_{-\infty}^{\infty}|H(f)|^2\,df} \tag{16.1–14}$$

is probably the best simple measure of bandwidth for real lowpass waveforms. These definitions take on added significance because for ΔT and ΔW so defined it is possible to prove the following celebrated principle:

*T_i and $(\Delta T_i/2)^2$ also have analogs in statistics (where $h_i(t)$, if positive, is analogous to a probability density) and are thus sometimes called the *mean* and *variance* (or *dispersion*) of $h_i(t)$.

UNCERTAINTY PRINCIPLE:

For any real waveform for which ΔT and ΔW of (16.1–13) and (16.1–14) exist,

$$\boxed{\Delta T\,\Delta W \geq \frac{1}{\pi}.} \qquad (16.1\text{–}15)$$

In words, ΔT and ΔW cannot simultaneously be arbitrarily *small*: A short duration implies a large bandwidth, and a small-bandwidth waveform must last a long time. In relation to measurements, (16.1–15) is often interpreted to imply that the uncertainty in the determination of a frequency is on the order of magnitude of the reciprocal of the time taken to measure it. Indeed, using the equivalences of relativity theory, (16.1–15) can be transformed into the *Heisenberg Uncertainty Principle* of wave mechanics, which asserts the impossibility of simultaneously specifying the precise position and conjugate momentum of a particle.

The proof of (16.1–15) is an interesting exercise in Fourier manipulations. In outline, the argument is as follows. Assume for simplicity (and without loss of generality, since the moment of inertia is a minimum about an axis through the center of gravity) that the time origin is chosen at the center of gravity of $h^2(t)$, so that the second term in (16.1–13) is zero. Then, since the Fourier transform of $dh(t)/dt = \dot{h}(t)$ is (from Table XIII.2) $j2\pi f\,H(f)$, we have, applying Parseval's Theorem,

$$(2\pi)^2 \int_{-\infty}^{\infty} f^2 |H(f)|^2\,df = \int_{-\infty}^{\infty} [\dot{h}(t)]^2\,dt. \qquad (16.1\text{–}16)$$

Combining this formula with (16.1–13) and (16.1–14) gives

$$(\pi\Delta T\Delta W)^2 = 4\frac{\int_{-\infty}^{\infty} t^2 h^2(t)\,dt \int_{-\infty}^{\infty} [\dot{h}(t)]^2\,dt}{\left(\int_{-\infty}^{\infty} h^2(t)\,dt\right)^2}. \qquad (16.1\text{–}17)$$

Next apply the Schwarz inequality (Example 14.A–1) to the numerator of (16.1–17) to obtain

$$\pi\Delta T\Delta W \geq \frac{2\left|\int_{-\infty}^{\infty} t h(t)\dot{h}(t)\,dt\right|}{\int_{-\infty}^{\infty} h^2(t)\,dt}. \qquad (16.1\text{–}18)$$

But the fraction on the right in (16.1–18) is identically equal to 1 (as can be seen by integrating either the numerator or the denominator by parts and noting that $h(t)$ must vanish rapidly as $t \to \pm\infty$ if the integrals in (16.1–13) and (16.1–14) are to exist as assumed), which gives the desired result.

It is interesting to explore the conditions under which the lower limit on the duration-bandwidth product can be achieved. To obtain equality in the Schwarz inequality as applied to (16.1–17) (see Problem 14.17) we must have

$$kth(t) = \dot{h}(t) \qquad (16.1\text{–}19)$$

or

$$\frac{\dot{h}(t)}{h(t)} = kt. \tag{16.1–20}$$

Integrating, we have

$$\ln h(t) = \frac{kt^2}{2} + \text{constant} \tag{16.1–21}$$

or

$$h(t) \sim e^{kt^2/2}. \tag{16.1–22}$$

For k negative this is an acceptable pulse-like waveform—the Gaussian function already discussed. Among all waveforms, then, *the Gaussian pulse has the smallest duration-bandwidth product* in the sense of (16.1–13) and (16.1–14).

Example 16.1–2

As an example of the Uncertainty Principle, consider the 3$^{\text{rd}}$-order Butterworth filter examined extensively in earlier chapters. The magnitude of the frequency response and the impulse response are

$$|H(f)|^2 = \frac{1}{1+(f/f_0)^6}$$

$$h(t) = 2\pi f_0 \left\{ e^{-2\pi f_0 t} + e^{-\pi f_0 t}\left[\frac{1}{\sqrt{3}} \sin\sqrt{3}\pi f_0 t - \cos\sqrt{3}\pi f_0 t\right]\right\} u(t).$$

These functions are sketched in Figure 16.1–2. We readily compute

$$\int_{-\infty}^{\infty} |H(f)|^2\, df = \int_0^{\infty} h^2(t)\, dt = \frac{2\pi f_0}{3}$$

$$\int_{-\infty}^{\infty} f^2 |H(f)|^2\, df = \frac{\pi f_0^3}{3}, \qquad \int_0^{\infty} t^2 h^2(t)\, dt = \frac{35}{36\pi f_0}$$

$$\int_0^{\infty} t h^2(t)\, dt = \frac{3}{4}$$

from which, by (16.1–13) and (16.1–14),

$$(\Delta W)^2 = 2 f_0^2, \qquad (\Delta T)^2 = \frac{37}{48(\pi f_0)^2}$$

and

$$(\Delta T \Delta W)^2 = \frac{37}{24}\frac{1}{\pi^2} > \frac{1}{\pi^2}$$

as required by the Uncertainty Principle. The relationships of ΔT and ΔW as measures of width to the actual impulse, step, and frequency response are shown in Figure 16.1–2.

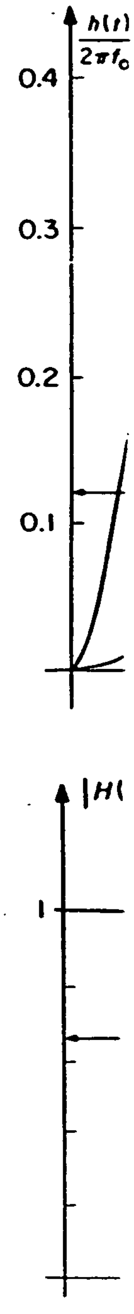

Figure 16.
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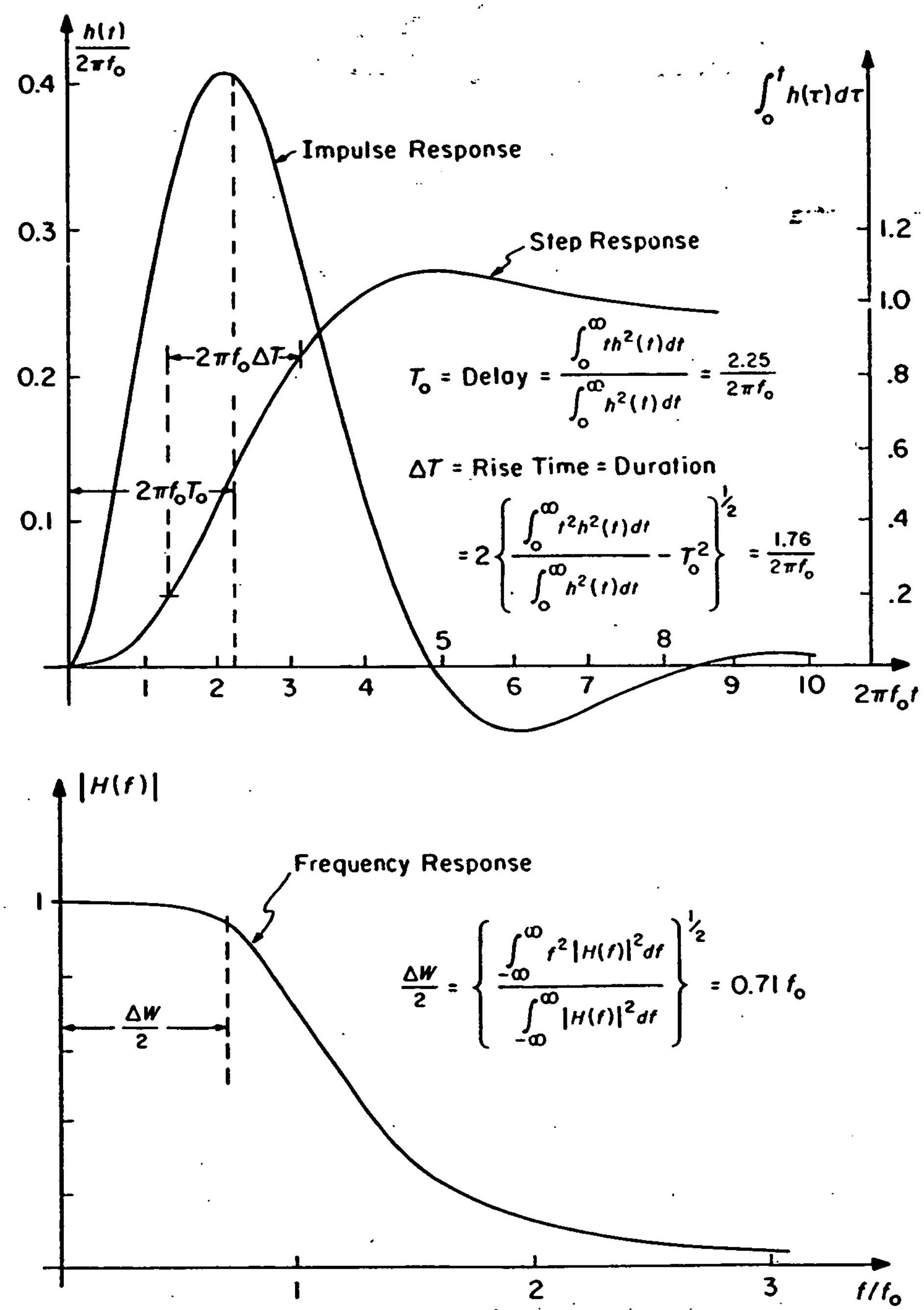

Figure 16.1–2. Impulse, step and frequency response of a 3rd-order Butterworth lowpass filter.

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Example 16.1–3

The Gaussian pulse cannot be the impulse response of a causal system, even with substantial delay (see Section 15.2). But it is easy to construct systems whose impulse responses approximate delayed Gaussian pulses. Consider, for example, an N-stage amplifier with the impulse response of each stage equal to

$$h(t) = \alpha\sqrt{N}e^{-\alpha\sqrt{N}t}u(t). \tag{16.1–23}$$

The frequency response of the cascade of N such stages is

$$H_N(f) = [H(f)]^N = \left(\frac{1}{1 + j2\pi f/\alpha\sqrt{N}}\right)^N. \tag{16.1–24}$$

We seek to determine the shape of $H_N(f)$ for large N. By taking logarithms,

$$\ln H_N(f) = -N\ln\left(1 + j\frac{2\pi f}{\alpha\sqrt{N}}\right) \tag{16.1–25}$$

and using the power series expansion $\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3}\cdots$, we have

$$\begin{aligned}\ln H_N &= -N\left[\frac{j2\pi f}{\alpha\sqrt{N}} - \frac{1}{2}\left(\frac{j2\pi f}{\alpha\sqrt{N}}\right)^2 + \frac{1}{3}\left(\frac{j2\pi f}{\alpha\sqrt{N}}\right)^3\cdots\right]\\ &\approx -j\frac{2\pi f}{\alpha}\sqrt{N} - \frac{1}{2}\left(\frac{2\pi f}{\alpha}\right)^2\end{aligned} \tag{16.1–26}$$

where the remaining terms vanish at least as fast as $1/\sqrt{N}$ for large N. Thus the frequency response of the cascade tends to

$$H_N(f) \approx e^{-\frac{1}{2}(2\pi f/\alpha)^2}e^{-j\sqrt{N}(2\pi f/\alpha)} \tag{16.1–27}$$

for large N, and the impulse response of the cascade tends to

$$h_N(t) \approx \frac{\alpha}{\sqrt{2\pi}}e^{-\frac{(t-\sqrt{N}/\alpha)^2}{2/\alpha^2}} \tag{16.1–28}$$

that is, a Gaussian pulse delayed by $\sqrt{N}/\alpha$.

This result is a very special case of a remarkable theorem*—the *Central Limit Theorem* of probability theory—which states in effect that, under very general conditions, the cascade of a large number of LTI systems will tend to have a Gaussian impulse response, almost independent of the characteristics of the systems cascaded! Sufficient conditions are that

1. the absolute third moments, $\int_{-\infty}^{\infty}|t|^3 h_i(t)\,dt$, exist for all the component systems and are uniformly bounded;

*See, for example, M. Fisz, *Probability Theory and Mathematical Statistics* (New York, NY: John Wiley, 1963). The version stated is a special case of Lyapunov's Theorem.

2. the durations, ΔT_i, of the component systems in the sense of (16.1–11) satisfy the relation $\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N}(\Delta T_i)^2 \neq 0$.

The first condition allows us to ignore for large N higher-order terms in an expansion such as (16.1–26), and the second guarantees that no finite subset of the component systems will dominate the result because the remainder all have relatively wide bandwidths. Given this theorem (which we shall not prove), it follows from Example 16.1–1 that the overall impulse response of N cascaded stages is approximately

$$h(t) \approx \frac{k}{\sqrt{2\pi}\Delta T} e^{-\frac{(t-T)^2}{2(\Delta T)^2}} \tag{16.1–29}$$

where T and ΔT are given by (16.1–4) and (16.1–12) respectively and

$$k = \prod_{i=1}^{N} \int_{-\infty}^{\infty} h_i(t)\,dt. \tag{16.1–30}$$

16.2 The Significance of the Uncertainty Principle; Pulse Resolution

The constraint imposed by the Uncertainty Principle on time waveforms and their transforms has an astonishingly wide domain of applications in addition to the rise-time/bandwidth context in which it was introduced—as the following examples illustrate.

Example 16.2–1

As we suggested in Section 14.4, the information waveform at an appropriate spot in a *pulse communication system* might appear as shown in Figure 16.2–1. In each successive time interval of duration T seconds a pulse of standardized shape and amplitude, such as the triangle shown below, is either presented or not presented accordingly as the symbol to be communicated in that interval is a "1" or a "0." The resulting waveform is then modulated on a carrier for transmission. Since the available bandwidth is usually limited for technical, economic, or legal reasons, the Uncertainty Principle sets a limit on the *bit rate* the system can achieve.

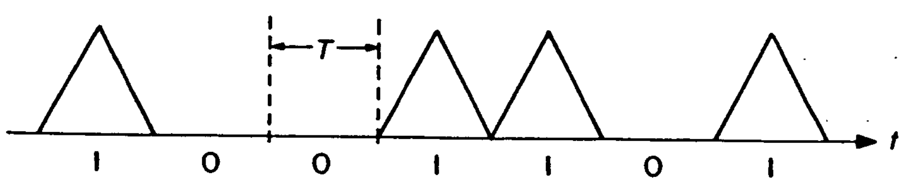

Figure 16.2–1. The information waveform in a pulse communication system.

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Example 16.2–2

A speech waveform contains quasi-periodicities, introduced by the glottal excitation and by resonances in the vocal tract, that are most readily studied in terms of the spectrum of the speech waveform. Of course, the waveform (and hence its quasi-periodic structure) is rapidly changing as the speaker moves from syllable to syllable (or more fundamentally from phoneme to phoneme)—indeed, it is precisely these changes that convey the meaning in the utterance and are thus of greatest interest. To follow these changes, we must compute the spectrum only over a time interval less than the phoneme duration. But the Uncertainty Principle tells us that our ability to resolve two frequency components will deteriorate inversely with the time taken to analyze them. Thus a compromise must be reached between our desires to follow rapid time changes and to explore fine details in the spectrum. Similar considerations apply to the spectral analysis of many phenomena—seismic records, weather patterns, ocean waves, econometric data, electroencephalograms and cardiograms, etc.

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Example 16.2–3

Designing a transmitting antenna for radio or radar systems usually involves either trying to maximize the field strength at some receiving site or trying to minimize the angular dimensions of a "pencil" or "fan" beam in which the radiated energy is to be concentrated. Since an increase in radiated intensity in one direction can be achieved only by reducing it in other directions, these goals are essentially the same. The maximization generally must be carried out subject to a constraint on the overall size (cost) of the antenna structure. On the other hand, the designer of a receiving antenna usually seeks either to maximize the effective cross-sectional area of the antenna for signals arriving from a given direction, or to narrow as much as possible the acceptance angle of the antenna so as to locate the direction of a source as precisely as possible. Again these goals are to be achieved for a fixed maximum aperture size of the actual antenna, and again these goals are essentially the same. Indeed, because a reciprocity principle applies to antennas that is similar to the reciprocity principle for circuits (in both cases, the principle follows directly from Maxwell's equations), the design problems for transmitting and receiving antennas are essentially identical—in both cases we seek to minimize a beam width subject to a constraint on aperture size. And since light is also an electromagnetic phenomenon, the same physical situation arises in the design of optical systems, such as microscopes or telescopes, to achieve maximum resolution. The connection between all of these problems and the Uncertainty Principle should be apparent once it is recognized that the distribution of energy in angle from a radiating antenna is essentially the magnitude of the spatial Fourier transform of the field strength across the aperture.* Reducing the aperture width necessarily broadens the beam angle, and the Uncertainty Principle sets a minimum on the product of beam angle and aperture width. (The value of this minimum does depend, however, on the wavelength or frequency of the radiated waveform.)

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*See, for example, R. Bracewell, *The Fourier Transform and Its Applications* (New York, NY: McGraw-Hill, 1965) Chap. 13.

Besides illustrating the wide significance of the Uncertainty Principle, these examples share another feature in common. In all of them a key problem is to design a pulse shape (or its dual, a spectral band) that meets three criteria simultaneously:

1. The spectrum of the pulse should essentially vanish for frequencies greater than some frequency W.
2. The pulse duration is to be reasonably close to the Uncertainty Principle limit.
3. The "tails" of the pulse must die away sufficiently rapidly that the tail of a large pulse will not "mask" or seriously distort another smaller pulse at an adjacent time instant; that is, we seek to be able to *resolve* two adjacent pulses even if one is much larger than the other. We must minimize as much as possible the effects of *interpulse* (or, in the dual situation, *interchannel*) *interference*.

No universally satisfactory solution to this problem is possible; the requirements are fundamentally contradictory. But there is one general principle that helps to suggest the nature of the compromises involved—the more slowly and smoothly a function changes, the more rapidly and precipitously its transform changes, and vice versa.

A formula that expresses this principle in a somewhat more quantitative fashion can be derived from Parseval's Theorem and the differentiation or the multiply-by-t properties:

$$(2\pi)^{2k}\int_{-\infty}^{\infty}|t|^{2k}|x(t)|^2\,dt=\int_{-\infty}^{\infty}\left|\frac{d^kX(f)}{df^k}\right|^2 df. \qquad (16.2\text{-}1)$$

Thus if all the derivatives of $X(f)$ through the $(n-1)^{\text{st}}$ are square-integrable but the n^{th} is not, we may in general conclude that $x(t)$ vanishes faster than $|t|^{-n+1/2}$ but no faster than $|t|^{-n-1/2}$. Indeed, for most ordinary square-integrable spectra, if $X(f)$ contains discontinuities but no worse singularities, then $n=1$ and $x(t)$ will typically vanish as $|t|^{-1}$. For example, the frequency response of an ideal lowpass filter contains such discontinuities, and the impulse response is $\frac{\sin 2\pi Wt}{\pi t}$, which vanishes as $|t|^{-1}$. Continuous spectra with discontinuous derivatives, such as $e^{-|f|}$ or the triangular spectrum of Fejér's kernel, imply $n=2$ and thus correspond to time functions that vanish more rapidly, in fact as $|t|^{-2}$. Other examples may be taken from Table XIII.1; thus, by the dual of this argument, the time function $e^{-\alpha t}u(t)$, being discontinuous, should have a spectrum falling off as $|f|^{-1}$, as indeed $(\alpha+j2\pi f)^{-1}$ does. Still other examples are provided by the following example.

Example 16.2–4

It is instructive to compare the spectra corresponding to each of the four pulses of Figure 16.2–2. The spectra may be computed in a straightforward manner; the results

are plotted in Figure 16.2–3. The square pulse, $x_1(t)$, and the triangular pulse, $x_2(t)$, have spectral tails falling off at 6 dB/octave (that is, as f^{-1}) and 12 dB/octave (that is, as f^{-2}) as expected; for the same value of T, the effective width of the triangular pulse is somewhat narrower, and hence the bandwidth of $X_2(f)$ is somewhat wider than that of $X_1(f)$. The very useful waveform $x_3(t)$ is called the *raised-cosine* (or *Hanning*) *pulse*. The tails of its Fourier transform decay at 18 dB/octave because the second derivative of $x_3(t)$ is square-integrable but discontinuous. Since $x_3(t)$ is even narrower than the triangular pulse $x_2(t)$, $X_3(f)$ has a still wider bandwidth. This illustrates a general principle: If the interval in which $x(t)$ is nonzero is restricted, more rapidly decaying tails on $X(f)$ by and large imply a corresponding increase in bandwidth. The fourth pulse, $x_4(t)$ (called the *Hamming pulse**), illustrates one of the many sorts of compromises that are often useful. In this case, by combining $x_3(t)$ and $x_1(t)$ with appropriate weights, the ~~height of th~~ose *side lobes* of $X_4(f)$ immediately adjacent to the *main lobe* can be substantially reduced at the expense of a slower rate of decay further out; since $x_4(t)$ is discontinuous, the tails of $X_4(f)$ must ultimately vanish as $|f|^{-1}$. The particular weighting in $x_4(t)$ was chosen to minimize the maximum side-lobe amplitude.

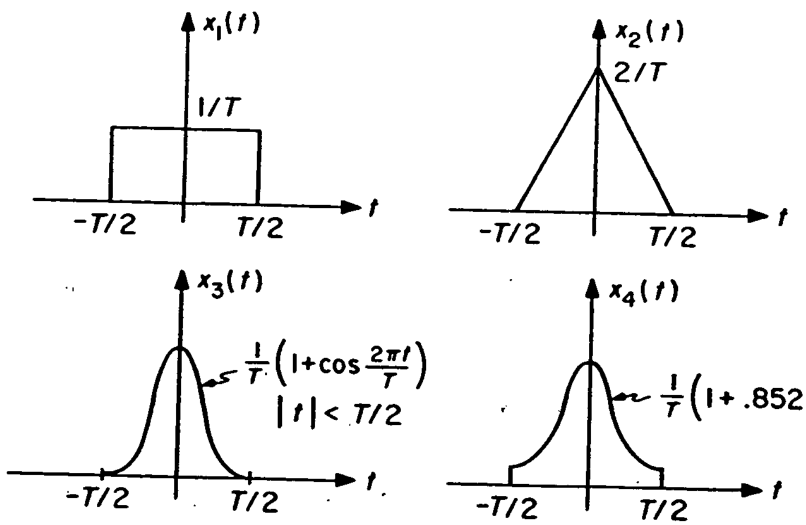

Figure 16.2–2. Four different pulse shapes.

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One application of the pulse-shape–spectral-shape relationships of Example 16.2–4 is provided by the pulse communication system described in Example 16.2–1. A succession of raised-cosine pulses of various amplitudes would have a slightly larger effective bandwidth than a similar train of square pulses or triangle pulses, but might be preferable in practice since the low tails of the raised-cosine spectrum would cause much less interference with other communication services occupying adjacent frequencies.

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*R. B. Blackman and J. W. Tukey, *The Measurement of Power Spectra* (New York, NY: Dover, 1958).

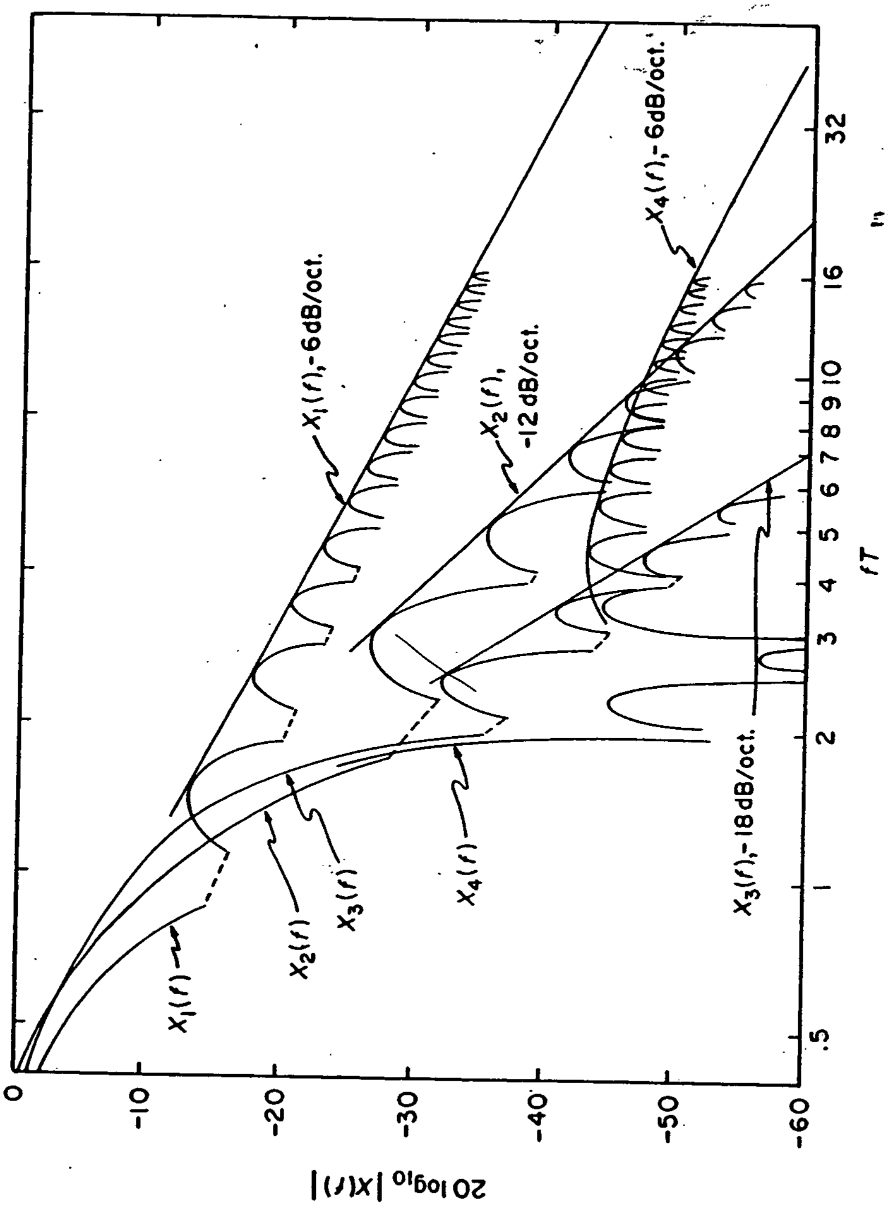


Figure 16.2-3. Spectra of the pulses of Figure 16.2-2.

A quite different application suggested by Example 16.2–3 is to the design of the large antennas used in radar or radio astronomy. Such antennas often employ parabolic reflectors to collect energy from a wide but finite aperture and focus it onto a single detector at the focal point of the "dish." If there is only a single active source (or radar reflector) in front of the antenna, then in general the output of the detector will be largest if this source is directly along the axis of the antenna, and it will fall off rapidly as the angle between the source and the antenna axis increases. The precise manner in which this *beam pattern* varies with the angle off axis depends on the weighting that the detector (for example, because of its shape) assigns to the signal coming from each element of the reflector surface. Often, areas near the edge of the reflector are intentionally less heavily weighted than areas near the center. The reason for such non-uniform weighting follows from the fact, noted in Example 16.2–3, that the beam pattern as a function of the angle, θ, is (for narrow beams) essentially the magnitude of the Fourier transform of the detector weighting as a function of position across the aperture. Thus, if the weighting is uniform across a finite aperature, the beam pattern is of the form $\frac{\sin k\theta}{k\theta}$ and the side lobes fall off only as $|\theta|^{-1}$. Lower side lobes can be obtained, at the expense of some increase in the width of the main beam, by an appropriate *tapered* weighting, which might be triangular or raised-cosine. The importance of such tapering is obvious once it is appreciated that in a radar, for example, the echo from a large nearby target may be more than 100 dB greater than that from a small remote target. Thus, even with a smooth raised-cosine tapering two targets might have to be separated by many beam widths before the echo from the weaker one would even be comparable with the side-lobe level of the stronger one.

Other applications will be discussed in the following chapters.

16.3 Summary

The Uncertainty Principle gives a lower bound on the product of the duration and bandwidth of any waveform. A related and almost equally powerful idea is that a rapidly decaying characteristic in one domain implies a high degree of smoothness in the other. Together, these concepts govern the design of waveforms and systems for an extraordinary range of applications—including the communication systems that will be the topic of our next chapter.

EXERCISES FOR CHAPTER 16

Exercise 16.1

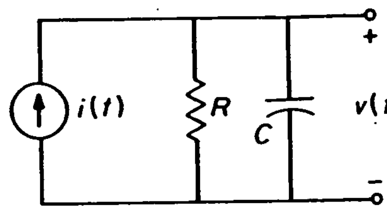
For an ideal filter with impulse response $h(t) = \dfrac{\sin 2\pi W t}{\pi t}$, show that:

a) The rise time of the step response as defined by (16.1–1) is $1/2W$.

b) The 10%-to-90% rise time is $0.446/W$.

Exercise 16.2

For the simple RC circuit shown below, prove that the product of the 10%-to-90% rise time and the 3-dB-down single-sided bandwidth is approximately 0.35. (Since the performance of many electronic amplifiers and other systems is often dominated by a single pole, this result provides an approximate rule-of-thumb for arbitrary lowpass systems that is often surprisingly accurate. For example, the 741 op-amp data sheet gives a unity-gain bandwidth of 10^6 Hz and a unity-gain rise time of 0.3 μsec.)



Exercise 16.3

Argue from the Paley-Wiener Theorem that no time function can be zero outside some finite time interval and also have a transform that is zero outside some finite frequency interval. Thus "duration" and "bandwidth" cannot simultaneously be interpreted in a strict sense.